

EFFECTIVE CLASSICAL HAMILTONIAN FROM PERTURBATIVELY DEFINED PATH INTEGRAL

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Introducing a perturbative definition, phase space path integrals can be calculated without slicing. This leads to a short-time expansion of the quantum-mechanical path amplitude, or a high-temperature expansion of the unnormalized density matrix, respectively. We use the proposed formalism to calculate the effective classical Hamiltonian for the harmonic oscillator.

1 Introduction

Path integrals are usually evaluated by time-slicing¹, since the continuum definition is mathematically problematic. This becomes obvious for physical systems with nontrivial metric, where reparametrization invariance seems to be violated.

An alternative method of calculation was proposed on the basis of a perturbative definition of path integrals in *configuration space*² motivated by perturbative procedures in quantum field theory. In this approach, the path integral of any system is expanded about the exactly known solution for the free particle in powers of the coupling constant of the potential.

The extension of this definition to functional integrals in *phase space* is presented in this paper by treating the complete Hamilton function as perturbation, which leads to a short-time expansion of the quantum-mechanical path amplitude. In Euclidean space, the density matrix is obtained as a high-temperature expansion. By simple resummation, this series can be turned into an expansion in powers of the coupling constant of the potential described above. As will be shown in the sequel, the knowledge of an exactly known nontrivial path integral as that of a free particle is, however, *not* required. Thus, the perturbative definition presented here is more general. It reproduces the expansion around the free particle by a simple re-summation.

The method is then applied to calculate the effective classical Hamiltonian of the harmonic oscillator $H_{\omega,\text{eff}}(p_0, x_0)$ by exactly summing up the perturbation series. This quantity is related to the quantum statistical partition

function via the classical looking phase space integral

$$Z_\omega = \int \frac{dx_0 dp_0}{2\pi\hbar} \exp \{-\beta H_{\omega, \text{eff}}(p_0, x_0)\}, \quad (1)$$

where $\beta = 1/k_B T$ is the inverse thermal energy.

2 Perturbative Definition of the Path Integral for the Density Matrix

Slicing the interval $[0, \hbar\beta]$ into $N + 1$ pieces of width $\varepsilon = \hbar\beta/(N + 1)$, the unnormalized density matrix can be expressed in the continuum limit by¹

$$\begin{aligned} \tilde{\varrho}(x_b, x_a) = & \lim_{N \rightarrow \infty} \prod_{n=1}^N \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} e^{ip_n(x_n - x_{n-1})} \right] \\ & \times \exp \left\{ -\varepsilon \sum_{n=1}^{N+1} H(p_n, x_n)/\hbar \right\}, \end{aligned} \quad (2)$$

where $x_a = x_0$ and $x_b = x_{N+1}$ are the fixed end points of the path. Expanding the last exponential in powers of ε/\hbar , we recognize that the zeroth order contribution to the density matrix (2) is an *infinite product of δ -functions* due to the identity

$$\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} e^{ip_n(x_n - x_{n-1})} = \delta(x_n - x_{n-1}). \quad (3)$$

This infinite product simply reduces to

$$\lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} dx_N \cdots dx_1 \delta(x_{N+1} - x_N) \cdots \delta(x_2 - x_1) \delta(x_1 - x_0) = \delta(x_b - x_a). \quad (4)$$

Note that this is the unperturbed contribution to the unnormalized density matrix (2) which was obtained without solving a nontrivial path integral. Thus, the phase space path integral for the unnormalized density matrix (2) can be perturbatively defined as

$$\begin{aligned} \tilde{\varrho}(x_b, x_a) = & \delta(x_b - x_a) + \sum_{n=1}^{\infty} \frac{(-1)^n}{\hbar^n n!} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \\ & \times \langle H(p(\tau_1), x(\tau_1)) \cdots H(p(\tau_n), x(\tau_n)) \rangle_0 \end{aligned} \quad (5)$$

with expectation values

$$\langle \cdots \rangle_0 = \lim_{N \rightarrow \infty} \prod_{n=1}^N \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \cdots e^{ip_n(x_n - x_{n-1})} \right]. \quad (6)$$

These expectation values are usually reexpressed by Feynman diagrams. This is possible for polynomial as well as nonpolynomial functions of momentum and position³.

3 Restricted Partition Function and Two-Point Correlations

The trace over the unnormalized density matrix (5) of our unperturbed system with vanishing Hamiltonian $H(p, x) = 0$ leads to a diverging partition function. It is also obvious that the classical partition function diverges with the phase space volume. The regularization of these divergences is possible by excluding from the phase space path integral the zero frequency fluctuations x_0 and p_0 of the Fourier decomposition of the periodic path $x(\tau)$ and momentum $p(\tau)$, respectively^{1,4}. Therefore we express the quantum statistical partition function of any system by

$$Z = \int \frac{dx_0 dp_0}{2\pi\hbar} Z^{p_0 x_0}, \quad (7)$$

where the restricted partition function was introduced as the Boltzmann factor of the effective classical Hamiltonian:

$$Z^{p_0 x_0} \equiv \exp \{ -\beta H_{\text{eff}}(p_0, x_0) \} = 2\pi\hbar \oint \mathcal{D}x \mathcal{D}p \delta(x_0 - \bar{x}) \delta(p_0 - \bar{p}) \\ \times \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \left[-i(p(\tau) - p_0) \frac{\partial}{\partial \tau} (x(\tau) - x_0) + H(p(\tau), x(\tau)) \right] \right\}, \quad (8)$$

with the measure

$$\oint \mathcal{D}x \mathcal{D}p = \lim_{N \rightarrow \infty} \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dx_n dp_n}{2\pi\hbar} \right] \quad (9)$$

and temporal mean values $\bar{x} = \int_0^{\hbar\beta} d\tau x(\tau) / \hbar\beta$ and $\bar{p} = \int_0^{\hbar\beta} d\tau p(\tau) / \hbar\beta$. As in the preceding section, the unperturbed system may have a vanishing Hamiltonian, i.e. $H = 0$. The calculation of the restricted partition function $Z_0^{p_0 x_0}$ for this system, i.e. evaluating the path integral (8) for $H = 0$, is in the same sense *trivial* as for its density matrix, since it also reduces to a cancellation of δ -functions and yields $Z_0^{p_0 x_0} = 1$.

In what follows, we concentrate on the correlation functions of position and momentum dependent quantities. For this purpose it is convenient to introduce the generating functional

$$Z_0^{p_0 x_0}[j, v] = 2\pi\hbar \oint \mathcal{D}x \mathcal{D}p \delta(x_0 - \bar{x}) \delta(p_0 - \bar{p}) \times \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \left[-i\tilde{p}(\tau) \frac{\partial}{\partial \tau} \tilde{x}(\tau) + j(\tau) \tilde{x}(\tau) + v(\tau) \tilde{p}(\tau) \right] \right\} \quad (10)$$

with abbreviations $\tilde{x}(\tau) = x(\tau) - x_0$ and $\tilde{p}(\tau) = p(\tau) - p_0$. The calculation yields

$$Z_0^{p_0 x_0}[j, v] = \exp \left\{ \frac{1}{\hbar^2} \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' j(\tau) G^{p_0 x_0}(\tau, \tau') v(\tau') \right\}, \quad (11)$$

where the periodic Green function is

$$\begin{aligned} G^{p_0 x_0}(\tau, \tau') &= -\frac{i}{2\beta} \{2(\tau - \tau') - \hbar\beta [\Theta(\tau - \tau') - \Theta(\tau' - \tau)]\} \\ &= \frac{2i}{\beta} \sum_{m=1}^{\infty} \frac{\sin \omega_m(\tau - \tau')}{\omega_m}. \end{aligned} \quad (12)$$

In the last line we have given the Fourier decomposition with respect to Matsubara frequencies $\omega_m = 2\pi m/\hbar\beta$ omitting the zero mode. Note that $G^{p_0 x_0}(\tau, \tau') = -G^{p_0 x_0}(\tau', \tau)$. These Green functions possess an interesting scaling property: Substituting $\bar{\tau} \equiv \tau/\beta$, the Green function becomes *independent* of β :

$$G^{p_0 x_0}(\bar{\tau}, \bar{\tau}') = -\frac{i}{2} \{2(\bar{\tau} - \bar{\tau}') - \hbar [\Theta(\bar{\tau} - \bar{\tau}') - \Theta(\bar{\tau}' - \bar{\tau})]\}. \quad (13)$$

Introducing expectation values as

$$\langle \cdots \rangle_0^{p_0 x_0} = 2\pi\hbar \oint \mathcal{D}x \mathcal{D}p \delta(x_0 - \bar{x}) \delta(p_0 - \bar{p}) \cdots \exp \left\{ \frac{1}{\hbar} \int_0^{\hbar\beta} d\tau i\tilde{p}(\tau) \frac{\partial}{\partial \tau} \tilde{x}(\tau) \right\}, \quad (14)$$

the two-point functions are obtained from functional (10) by performing appropriate functional derivatives with respect to $j(\tau)$ and $v(\tau)$, respectively:

$$\langle \tilde{x}(\tau) \tilde{x}(\tau') \rangle_0^{p_0 x_0} = 0, \quad (15)$$

$$\langle \tilde{x}(\tau) \tilde{p}(\tau') \rangle_0^{p_0 x_0} = G^{p_0 x_0}(\tau, \tau'), \quad (16)$$

$$\langle \tilde{p}(\tau) \tilde{p}(\tau') \rangle_0^{p_0 x_0} = 0. \quad (17)$$

The result is that only *mixed* position-momentum correlations do not vanish.

4 Perturbative Expansion for Effective Classical Hamiltonian

Expanding the restricted partition function (8) about a vanishing Hamiltonian,

$$Z^{p_0 x_0} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\hbar^n n!} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \times \langle H(p(\tau_1), x(\tau_1)) \cdots H(p(\tau_n), x(\tau_n)) \rangle_0^{p_0 x_0}, \quad (18)$$

rewriting this into a cumulant expansion, and utilizing the relation (8) between restricted partition function and effective classical Hamiltonian, we obtain

$$H_{\text{eff}}(p_0, x_0) = \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\hbar^n n!} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \times \langle H(p(\tau_1), x(\tau_1)) \cdots H(p(\tau_n), x(\tau_n)) \rangle_{0,c}^{p_0 x_0}. \quad (19)$$

Using Wick's rule, all correlation functions can be expressed in terms of products of two-point functions. Since only mixed two-point functions (12) can lead to nonvanishing contributions to the effective classical Hamiltonian, we use the rescaled version (13) of the Green function. The scaling transformation gives a factor β from each of the n integral measures. Thus the expansion (19) is a *high-temperature* expansion of the effective classical Hamiltonian:

$$H_{\text{eff}}(p_0, x_0) = \sum_{n=1}^{\infty} \beta^{n-1} \frac{(-1)^{n+1}}{\hbar^n n!} \int_0^{\hbar} d\bar{\tau}_1 \cdots \int_0^{\hbar} d\bar{\tau}_n \times \langle H(p(\bar{\tau}_1), x(\bar{\tau}_1)) \cdots H(p(\bar{\tau}_n), x(\bar{\tau}_n)) \rangle_{0,c}^{p_0 x_0}. \quad (20)$$

For the following considerations it is useful to assume the Hamilton function to be of standard form: $H(p(\bar{\tau}), x(\bar{\tau})) = p^2(\bar{\tau})/2M + gV(x(\bar{\tau}))$. Here we have introduced the coupling constant g of the potential. Defining the functionals $a[p] = \int_0^{\hbar} d\bar{\tau} p^2(\bar{\tau})/2M$ and $b[x] = \int_0^{\hbar} d\bar{\tau} V(x(\bar{\tau}))$, the high-temperature expansion (20) is expressed as

$$H_{\text{eff}}(p_0, x_0) = \sum_{n=1}^{\infty} \beta^{n-1} \frac{(-1)^{n+1}}{n! \hbar^n} \sum_{k=0}^n g^k \binom{n}{k} \langle a^{n-k}[p] b^k[x] \rangle_{0,c}^{p_0 x_0}. \quad (21)$$

In the sequel we point out how this high-temperature expansion is connected with an expansion in powers of the coupling constant g of the potential.

5 High-Temperature Versus Weak-Coupling Expansion

In the preceding section it was shown that the perturbative expansion about a vanishing Hamiltonian leads to a perturbative series in powers of the inverse temperature in a natural manner. Now we elaborate the relation to a perturbative expansion in powers of the coupling constant g of the potential. Changing the order of summation in Eq. (21), one obtains

$$H_{\text{eff}}(p_0, x_0) = \sum_{k=0}^{\infty} g^k \sum_{n=0}^{\infty} \beta^{n+k-1} \binom{n+k}{k} \frac{(-1)^{n+k+1}}{(n+k)! \hbar^{n+k}} \langle a^n[p] b^k[x] \rangle_{0,c}^{p_0 x_0} + \frac{1}{\beta}, \quad (22)$$

which is rewritten after explicitly evaluating the $(n = 0)$ - and $(k = 0)$ -contributions:

$$\begin{aligned} H_{\text{eff}}(p_0, x_0) &= \frac{p_0^2}{2M} + gV(x_0) + \frac{1}{\beta} \sum_{k=1}^{\infty} g^k \sum_{n=1}^{\infty} \frac{(-1)^{n+k+1}}{n! k! \hbar^{n+k} (2M)^n} \\ &\quad \times \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_k \int_0^{\hbar\beta} d\tau_{k+1} \cdots \int_0^{\hbar\beta} d\tau_{k+n} \\ &\quad \times \langle V(x(\tau_1)) \cdots V(x(\tau_k)) p^2(\tau_{k+1}) \cdots p^2(\tau_{k+n}) \rangle_{0,c}^{p_0 x_0} \end{aligned} \quad (23)$$

In this expression, we have inversed the scaling transformation and used that $\int_0^{\hbar\beta} d\tau \langle p^2(\tau) \rangle_{0,c}^{p_0 x_0} = \hbar\beta p_0^2$ and $\int_0^{\hbar\beta} d\tau \langle V(x(\tau)) \rangle_{0,c}^{p_0 x_0} = \hbar\beta V(x_0)$. Higher-order expectations of functions only depending on x or p are zero. This is due to the vanishing of expectations of functions of \tilde{x} or \tilde{p} which are decomposed into products of two-point functions (15) and (17). All other possible contributions are non-connected.

Note that the expansion (23) is equal to the expansion about the free particle

$$\begin{aligned} H_{\text{eff}}(p_0, x_0) &= \frac{p_0^2}{2M} + gV(x_0) + \frac{1}{\beta} \sum_{k=1}^{\infty} g^k \frac{(-1)^{k+1}}{k! \hbar^k} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_k \\ &\quad \times \langle V(x(\tau_1)) \cdots V(x(\tau_k)) \rangle_{\text{free},c}^{x_0}, \end{aligned} \quad (24)$$

where the new cumulants are formed from expectation values

$$\begin{aligned} \langle \cdots \rangle_{\text{free}}^{x_0} &= 2\pi\hbar \oint \mathcal{D}x \mathcal{D}p \delta(x_0 - \bar{x}) \delta(p_0 - \bar{p}) \cdots \\ &\quad \times \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \left[-i(p(\tau) - p_0) \frac{\partial}{\partial \tau} (x(\tau) - x_0) + \frac{1}{2M} (p(\tau) - p_0)^2 \right] \right\}. \end{aligned} \quad (25)$$

6 Effective Classical Hamiltonian of Harmonic Oscillator

In this section, we calculate the effective classical Hamiltonian for the harmonic oscillator

$$H_\omega(p, x) = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 x^2 \quad (26)$$

by an exact resummation of the high-temperature expansion (21). For systematically expressing the terms of this expansion, it is useful to introduce the following Feynman rules:

$$\bar{\tau}_1 \text{ --- } \bar{\tau}_2 \equiv \langle p(\bar{\tau}_1)p(\bar{\tau}_2) \rangle_0^{p_0 x_0} = p_0^2, \quad (27)$$

$$\bar{\tau}_1 \text{ --- } \bar{\tau}_2 \equiv \langle x(\bar{\tau}_1)x(\bar{\tau}_2) \rangle_0^{p_0 x_0} = x_0^2, \quad (28)$$

$$\bar{\tau}_1 \text{ --- } \bar{\tau}_2 \equiv \langle x(\bar{\tau}_1)p(\bar{\tau}_2) \rangle_0^{p_0 x_0} = G^{p_0 x_0}(\bar{\tau}_1, \bar{\tau}_2) + x_0 p_0, \quad (29)$$

$$\bar{\tau}_1 \text{ --- } \bar{\tau}_2 \equiv \langle p(\bar{\tau}_1)x(\bar{\tau}_2) \rangle_0^{p_0 x_0} = -G^{p_0 x_0}(\bar{\tau}_1, \bar{\tau}_2) + p_0 x_0, \quad (30)$$

$$\bar{\tau} \text{ --- } \star \equiv \langle p(\bar{\tau}) \rangle_0^{p_0 x_0} = p_0, \quad (31)$$

$$\bar{\tau} \text{ --- } \star \equiv \langle x(\bar{\tau}) \rangle_0^{p_0 x_0} = x_0, \quad (32)$$

$$\bullet \equiv \int_0^{\hbar} d\bar{\tau}, \quad (33)$$

where the current-like expectations in (31) and (32) arise from $\langle \tilde{p}(\bar{\tau}) \rangle_0^{p_0 x_0} = 0$ and $\langle \tilde{x}(\bar{\tau}) \rangle_0^{p_0 x_0} = 0$, respectively. In order to simplify the investigation of the expectation values in the high-temperature expansion of the effective classical Hamiltonian (20), we also define operational subgraphs which are useful to systematically construct Feynman diagrams:

$$\text{---}\bullet\text{---} \equiv \frac{1}{2M\hbar} \int_0^{\hbar} d\bar{\tau} p^2(\bar{\tau}), \quad (34)$$

$$\text{---}\bullet \equiv \frac{1}{2\hbar} M\omega^2 \int_0^{\hbar} d\bar{\tau} x^2(\bar{\tau}). \quad (35)$$

Feynman diagrams are built up by attaching the legs of such subgraphs to others or by connecting a leg with a suitable current. Note that only combinations of different types of subgraphs lead to non-vanishing contributions, since the connection of subgraphs of same type,

$$\text{---}\bullet\text{---}\bullet\text{---}, \quad \text{---}\bullet\text{---}\bullet, \quad (36)$$

sets up a new subgraph which contains a propagator (27) or (28), respectively. These propagators, however, are independent of τ , such that the τ -integrals related to the vertices in these subgraphs are trivial. Thus there does not

really exist a connection between these vertices and the propagators (27) and (28) can be expressed by the currents (31) and (32):

$$\overline{\tau}_1 \text{ --- } \overline{\tau}_2 = \overline{\tau}_1 \text{ --- } \star \text{ --- } \overline{\tau}_2, \quad (37)$$

$$\overline{\tau}_1 \text{ --- } \overline{\tau}_2 = \overline{\tau}_1 \text{ --- } \star \text{ --- } \overline{\tau}_2. \quad (38)$$

Therefore, for $n > 1$, connected diagrams containing propagators of type (27) or (28) *must* break into non-connected parts which we are not interested in. Analytically, this is seen by considering for example

$$\langle x(\overline{\tau}_1)x(\overline{\tau}_2) \rangle_0^{p_0 x_0} = \langle \tilde{x}(\overline{\tau}_1)\tilde{x}(\overline{\tau}_2) \rangle_0^{p_0 x_0} + \langle x(\overline{\tau}_1) \rangle_0^{p_0 x_0} \langle x(\overline{\tau}_2) \rangle_0^{p_0 x_0}. \quad (39)$$

The first term on the right-hand side vanishes due to Eq. (15), while the second simply yields x_0^2 , which proves Eq. (30). This means that only Feynman diagrams which consist of a mixture of subgraphs (34) and (35) contribute to the effective classical Hamiltonian. To illustrate this, we discuss the first and second order of expansion (20) in more detail.

The Feynman diagrams of the first-order contribution to the effective classical Hamiltonian are simply constructed from the subgraphs

$$\begin{aligned} H_{\omega, \text{eff}}^{(1)}(p_0, x_0) &\propto \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \\ &= \frac{1}{2M\hbar} \text{---} \bullet \text{---} + \frac{1}{2\hbar} M\omega^2 \text{---} \bullet \text{---} = \frac{1}{2M\hbar} \star \text{---} \bullet \text{---} \star + \frac{1}{2\hbar} M\omega^2 \star \text{---} \bullet \text{---} \star \\ &= \frac{p_0^2}{2M} + \frac{1}{2} M\omega^2 x_0^2, \end{aligned} \quad (40)$$

where we have used the identities (37) and (38) in the second expression of the second line. Note that the first order term (40) obviously reproduces the classical Hamiltonian. This is the consequence of the high-temperature expansion (21), since only the first-order contribution is nonzero in the limit $\beta = 1/k_B T \rightarrow 0$. The second-order contribution reads

$$\begin{aligned} H_{\omega, \text{eff}}^{(2)}(p_0, x_0) &\propto (\text{---} \bullet \text{---} + \text{---} \bullet \text{---}) (\text{---} \bullet \text{---} + \text{---} \bullet \text{---}) \\ &= -\frac{\omega^2}{8\hbar^2\beta} \left(8 \star \text{---} \bullet \text{---} \bullet \text{---} \star + 4 \text{---} \bullet \text{---} \bullet \text{---} \right). \end{aligned} \quad (41)$$

The chain diagram is zero, while the loop diagram has the value $-\hbar^4 \zeta(2)/2\pi^2$, where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (42)$$

is the ζ -function. Thus we obtain $H_{\omega,\text{eff}}^{(2)}(p_0, x_0) = \beta \hbar^2 \omega^2 \zeta(2)/4\pi^2$. This second-order contribution (41) shows the characteristic types of Feynman diagrams in each order $n > 1$ of the expansion (20) for the harmonic oscillator: chain and loop diagrams. In order to calculate the n th-order contribution, we must evaluate these diagrams more general. By constructing Feynman diagrams from the product of n sums of subgraphs,

$$H_{\omega,\text{eff}}^{(n)}(p_0, x_0) \propto \underbrace{(\text{---}\sim\text{---} + \text{---}\bullet\text{---}) (\text{---}\sim\text{---} + \text{---}\bullet\text{---}) \cdots (\text{---}\sim\text{---} + \text{---}\bullet\text{---})}_{n \text{ times}}, \quad (43)$$

it turns out that only following chain and loop diagrams contribute:

$$\begin{array}{c} \star \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\star, \\ \star \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\star, \\ \star \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\bullet \text{---}\sim\text{---}\star, \end{array} \quad \text{and} \quad \text{a circular loop diagram with } n \text{ vertices.} \quad (44)$$

The evaluation of the chain diagrams is easily done and yields zero. An explicit calculation in Fourier space shows that there occur Kronecker- δ 's δ_{m0} . Since the Matsubara sum of the Green function (12)) does not contain the zero mode $m = 0$, all chain diagrams are zero.

Determining the values of loop diagrams is more involved. It is obvious that loop diagrams can only be constructed in *even* order ($n = 2, 4, 6, \dots$), since for a loop diagram with mixed propagators (29) or (30) pairs of different subgraphs (34) and (35) are necessary. Thus we have found the result that *odd* orders of expansion (21) *vanish*, and only loop diagrams for $n \in \{2, 4, 6, \dots\}$ must be calculated. Evaluating loop diagrams of n th order in Fourier space is straightforward and entails

$$\text{a circular loop diagram with } n \text{ vertices} = 2(-1)^k \left(\frac{\hbar^2}{2\pi} \right)^{2k} \zeta(2k), \quad (45)$$

where $k = n/2$. The high-temperature expansion for the effective Hamiltonian of the harmonic oscillator can thus be written as

$$H_{\omega,\text{eff}}(p_0, x_0) = \frac{p_0^2}{2M} + \frac{1}{2}M\omega^2 x_0^2 + \sum_{k=1}^{\infty} \beta^{2k-1} \frac{(-1)^{k+1}}{k} \left(\frac{\hbar\omega}{2\pi} \right)^{2k} \zeta(2k). \quad (46)$$

Substituting the ζ -function by its definition (42) and exchanging the summations, the last term in Eq. (46) can be expressed as a logarithm

$$\sum_{k=1}^{\infty} \beta^{2k-1} \frac{(-1)^{k+1}}{k} \left(\frac{\hbar\omega}{2\pi} \right)^{2k} \zeta(2k) = \frac{1}{\beta} \ln \left(\prod_{n=1}^{\infty} \left[1 + \frac{\hbar^2 \beta^2 \omega^2}{4\pi^2 n^2} \right] \right). \quad (47)$$

Applying the relation

$$\frac{1}{z} \sinh z = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2 \pi^2} \right), \quad (48)$$

we find the more familiar form of the effective classical Hamiltonian for a harmonic oscillator

$$H_{\omega, \text{eff}}(p_0, x_0) = \frac{p_0^2}{2M} + \frac{1}{2} M \omega^2 x_0^2 - \frac{1}{\beta} \ln \frac{\hbar \omega \beta}{2 \sinh \hbar \omega \beta / 2}. \quad (49)$$

Performing the x_0 - and p_0 -integrations in Eq. (1), we obtain the well-known form of the partition function of the harmonic oscillator $Z_\omega = 1/\sinh \hbar \omega \beta$.

7 Summary

We have used a perturbative definition of the path integral in phase space representation which reproduces the effective classical Hamiltonian for the harmonic oscillator. Our procedure is an alternative way to evaluate path integrals: The unperturbed system is trivial and the calculation of appropriate Feynman diagrams is simple. Furthermore it turns out that the perturbative expansion for the effective classical Hamiltonian is identical to the high-temperature expansion.

Acknowledgements

I deeply thank Professor H. Kleinert for many exciting discussions, useful hints, and expert advice in the surrounding of my PhD studies. Also, I'm indebted to Dr. A. Pelster and Dr. A. Chervyakov for conversations regarding the perturbatively defined path integral. Finally, I'm grateful to the Studienstiftung des deutschen Volkes for support.

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